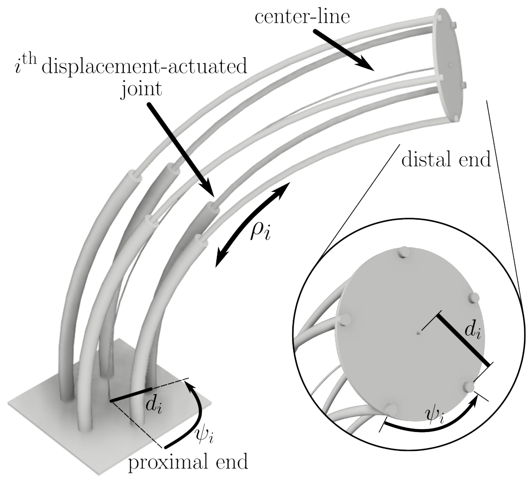
# Clarke Transform in a Nutshell

Cheat Sheet for mathematical modelling of continuum robots with $n$ symmetrical arranged displacement-actuated joints.

## General setup and idea

Schematics of a displacement-actuated robot:




Its sufficient smooth center-line has length $l$. Each of the displacement-actuated joint $\rho_i$ are equal distributed described by $\psi_i = 2\pi(i-1)/n$ and $d_i = d > 0$.

Commutative diagram-like overview:

$$\begin{tikzcd}
(\rho_{\mathrm{Re}}, \rho_{\mathrm{Im}}) \arrow[rr, "2\,\mathrm{dof}"] & \text{easy} & (\rho^*_{\mathrm{Re}}, \rho^*_{\mathrm{Im}}) \arrow[dd, "\boldsymbol{M}_{\mathcal{P}}^{-1}"] \\
 & \text{hard} & \\
\rho_i,\ i = 1, 2, \dots, n \arrow[uu, "\boldsymbol{M}_{\mathcal{P}}"] \arrow[rr, "n\,\mathrm{dof}", "\sum \rho_i = 0"'] & & \rho^*_i,\ i = 1, 2, \dots, n
\end{tikzcd}$$

To circumvent the explicit consideration of the displacement constraint, *i.e.*, $\sum \rho_i = 0$, approaches and methods should be considered on the two-dimensional manifold embedded in the $n$-dimensional joint space. For the transform, linear maps $\boldsymbol{M}_{\mathcal{P}}$ and $\boldsymbol{M}_{\mathcal{P}}^{-1}$ are used and any output denoted by $(\rho^*_{\mathrm{Re}}, \rho^*_{\mathrm{Im}})$ can subsequently mapped back leading to $\rho^*$.

## Clarke transform and Clarke coordinates

Generalized Clarke transformation matrix:

$$\boldsymbol{M}_{\mathcal{P}} = \frac{2}{n}\begin{bmatrix} \cos(0) & \cos\left(2\pi\frac{1}{n}\right) & \cdots & \cos\left(2\pi\frac{n-1}{n}\right) \\ \sin(0) & \sin\left(2\pi\frac{1}{n}\right) & \cdots & \sin\left(2\pi\frac{n-1}{n}\right) \end{bmatrix}$$

Generalized inverse Clarke transformation matrix:

$$\boldsymbol{M}_{\mathcal{P}}^{-1} = \begin{bmatrix} \cos(0) & \sin(0) \\ \cos\left(2\pi\frac{1}{n}\right) & \sin\left(2\pi\frac{1}{n}\right) \\ \vdots & \vdots \\ \cos\left(2\pi\frac{n-1}{n}\right) & \sin\left(2\pi\frac{n-1}{n}\right) \end{bmatrix}$$

Representation of $i^{\text{th}}$ displacement:

$$\rho_i = \rho_{\mathrm{Re}}\cos\psi_i + \rho_{\mathrm{Im}}\sin\psi_i$$

Kirchhoff's rule is the displacement constraint:

$$\textstyle\sum_{i=1}^{n} \rho_i = 0$$

Set of displacement-actuated joint values:

$$\boldsymbol{\rho} = \begin{bmatrix} \rho_1 & \rho_2 & \cdots & \rho_{n-1} & \rho_n \end{bmatrix}^\top$$

Clarke coordinates as vector:

$$\overline{\boldsymbol{\rho}} = \begin{bmatrix} \rho_{\mathrm{Re}} & \rho_{\mathrm{Im}} \end{bmatrix}^\top$$

The Clarke transform of $\boldsymbol{\rho}$ and $\overline{\boldsymbol{\rho}}$, respectively:

$$\overline{\boldsymbol{\rho}} = \boldsymbol{M}_{\mathcal{P}}\boldsymbol{\rho} \quad \text{and} \quad \boldsymbol{\rho} = \boldsymbol{M}_{\mathcal{P}}^{-1}\overline{\boldsymbol{\rho}}$$

Two-dimensional Manifold embedded in the joint space:

$$\mathbf{Q} = \left\{ (\rho_1, \cdots, \rho_n) \in \mathbb{R}^n \;\middle|\; \forall i \in [1, n] \subset \mathbb{N} : \begin{array}{l} \rho_i = \rho_{\mathrm{Re}}\cos\psi_i + \rho_{\mathrm{Im}}\sin\psi_i \;\wedge \\ (\rho_{\mathrm{Re}}, \rho_{\mathrm{Im}}) \in \mathbb{R}^2 \end{array} \right\}$$

## Properties

Linearity of $\boldsymbol{M}_{\mathcal{P}}$ and $\boldsymbol{M}_{\mathcal{P}}^{-1}$:

$$\boldsymbol{M}_{\mathcal{P}}\sum_i \boldsymbol{\rho}_i = \sum_i \boldsymbol{M}_{\mathcal{P}}\boldsymbol{\rho}_i = \sum_i \overline{\boldsymbol{\rho}}_i$$

$$\boldsymbol{M}_{\mathcal{P}}^{-1}\sum_i \overline{\boldsymbol{\rho}}_i = \sum_i \boldsymbol{M}_{\mathcal{P}}^{-1}\overline{\boldsymbol{\rho}}_i = \sum_i \boldsymbol{\rho}_i$$

$\boldsymbol{M}_{\mathcal{P}}^{-1}$ is the right-inverse of $\boldsymbol{M}_{\mathcal{P}}$:

$$\boldsymbol{M}_{\mathcal{P}}\boldsymbol{M}_{\mathcal{P}}^{-1} = \boldsymbol{I}_{2\times 2} \quad \text{and} \quad \boldsymbol{M}_{\mathcal{P}}^{-1}\boldsymbol{M}_{\mathcal{P}} \neq \boldsymbol{I}_{n\times n}$$

Toeplitz matrix:

$$\boldsymbol{M}_{\mathcal{P}}^{-1}\boldsymbol{M}_{\mathcal{P}} = \left(\frac{2}{n}\cos\left(2\pi\frac{i-j}{n}\right)\right)_{i,j} \in \mathbb{R}^{n\times n}$$

Toeplitz matrix is an idempotent matrix:

$$\left(\boldsymbol{M}_{\mathcal{P}}^{-1}\boldsymbol{M}_{\mathcal{P}}\right)^k = \boldsymbol{M}_{\mathcal{P}}^{-1}\boldsymbol{M}_{\mathcal{P}} \quad \text{for } k > 0$$

Toeplitz matrix is singular:

$$\det \boldsymbol{M}_{\mathcal{P}}^{-1}\boldsymbol{M}_{\mathcal{P}} = 0$$

Transpose:

$$\boldsymbol{M}_{\mathcal{P}}^\top = \frac{2}{n}\boldsymbol{M}_{\mathcal{P}}^{-1}$$

Vanishing bias term:

$$\boldsymbol{M}_{\mathcal{P}}\mathbf{1}_{1\times n} = \mathbf{0}_{1\times 2},$$

where $\mathbf{1}_{1\times n}$ has ones everywhere and $\mathbf{0}_{1\times 2}$ has zeros everywhere.

Selecting a mode:

$$\boldsymbol{M}_{\mathcal{P}}\mathbb{1}^{(k)}_{1\times n} = \begin{bmatrix} \cos\left(2\pi\frac{k-1}{n}\right) & \sin\left(2\pi\frac{k-1}{n}\right) \end{bmatrix}^\top,$$

where $\mathbb{1}^{(k)}_{1\times n}$ is a one-hot vector defined by the $k^{\text{th}}$ element to be a one, whereas the all other elements are zero.

## Properties cont'd

The sum of squares:

$$\rho_{\mathrm{Re}}^2 + \rho_{\mathrm{Im}}^2 = \overline{\boldsymbol{\rho}}^\top \overline{\boldsymbol{\rho}} = \boldsymbol{\rho}^\top \boldsymbol{M}_{\mathcal{P}}^\top \boldsymbol{M}_{\mathcal{P}} \boldsymbol{\rho}$$

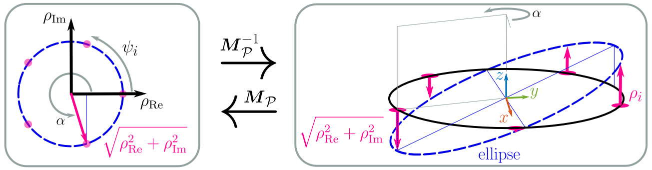
Scaled magnitude:

$$\overline{\boldsymbol{\rho}}^\top \overline{\boldsymbol{\rho}} = \frac{2}{n} \boldsymbol{\rho}^\top \boldsymbol{\rho}$$

Transform a unit circle:

$$\boldsymbol{M}_{\mathcal{P}}^{-1} \left[\cos(\alpha) \quad \sin(\alpha)\right]^\top = \left(\cos(\psi_i - \alpha)\right)_{i,1}$$

Visual aid and geometric interpretation:




Tracing the tip of all possible displacement parameterized by $\psi \in [0, 2\pi)$ creates an ellipse. Its semi-major and semi-minor axes are $\sqrt{d^2 + \rho_{\mathrm{Re}}^2 + \rho_{\mathrm{Im}}^2}$ and $d$ length, respectively. The maximum displacement achievable coincides with the angle $\alpha$.

Normalized displacement-actuated joints:

$$\widehat{\boldsymbol{\rho}} = \frac{\boldsymbol{\rho}}{\sqrt{\boldsymbol{\rho}^\top \boldsymbol{\rho}}}$$

To avoid singularities, add a sufficient small $\epsilon > 0$ with $\epsilon^2 \approx 0$ to the magnitude, *i.e.*, $\sqrt{\boldsymbol{\rho}^\top \boldsymbol{\rho}} + \epsilon$

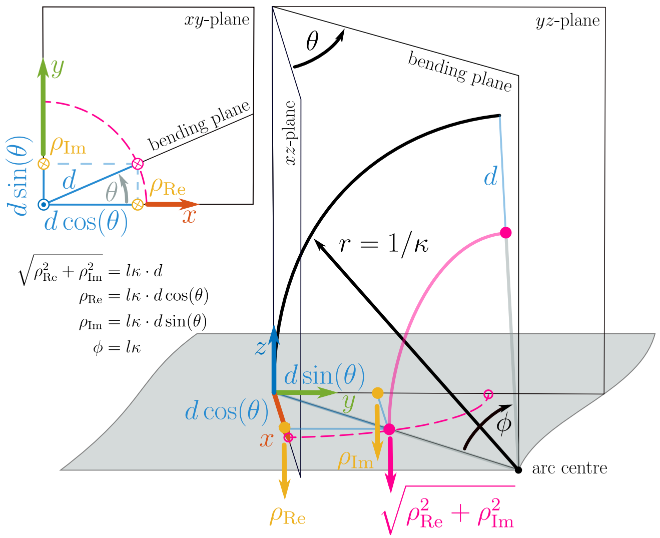
## Trigonometric Identities

Useful trigonometric identities for $\psi_i = \frac{2\pi}{n}(i-1)$:

$$\sum_{i=1}^{n} \sin(\psi_i) = 0, \quad \sum_{i=1}^{n} \sin^2(\psi_i) = \frac{n}{2},$$
$$\sum_{i=1}^{n} \cos(\psi_i) = 0, \quad \sum_{i=1}^{n} \cos^2(\psi_i) = \frac{n}{2},$$
$$\text{and} \quad \sum_{i=1}^{n} \sin(\psi_i)\cos(\psi_i) = 0$$

## Arc space and virtual displacement

Physical interpretation of the Clarke coordinates:




The magenta line lies within the bending plane. The length difference to the arc length is the virtual displacement. The yellow arrows are the projected virtual displacements and lie within the respective projected plane corresponding to $xz$-plane and $yz$-plane of the base.

Relation to arc space:

$$\boldsymbol{M}_{\mathcal{P}} \boldsymbol{\rho} = \left[\rho_{\mathrm{Re}} \quad \rho_{\mathrm{Im}}\right]^\top = ld \left[\kappa\cos(\theta) \quad \kappa\sin(\theta)\right]^\top$$

The design parameters, *i.e.*, segment length $l$ and joint location $(\phi_i, d_i)$, are removed, *i.e.*,

$$\begin{bmatrix} \kappa\cos(\theta) \\ \kappa\sin(\theta) \end{bmatrix} = \underbrace{1/l}_{\text{removes } l} \; \overbrace{\boldsymbol{M}_{\mathcal{P}}}^{\text{removes } \psi_i} \; \underbrace{\operatorname{diag}(1/d_i)}_{\text{removes } d_i} \boldsymbol{\rho}.$$

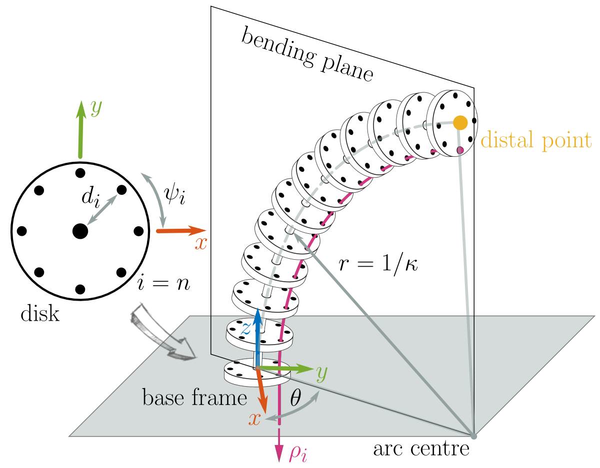
For its inverse, the design parameters are added, *i.e.*,

$$\boldsymbol{\rho} = \underbrace{l}_{\text{adds } l} \; \overbrace{\operatorname{diag}(d_i)}^{\text{adds } d_i} \; \underbrace{\boldsymbol{M}_{\mathcal{P}}^{-1}}_{\text{adds } \psi_i} \begin{bmatrix} \kappa\cos(\theta) \\ \kappa\sin(\theta) \end{bmatrix}.$$

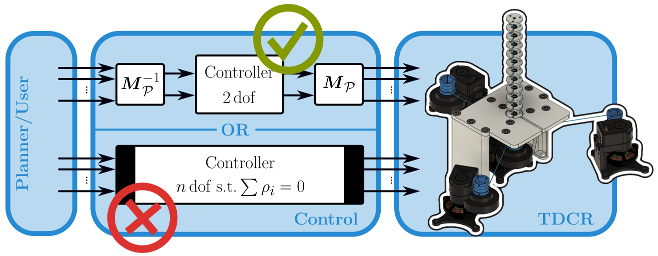
For both formulation, the assumption $d_i = d$ has been removed.

## Application to tendon-driven continuum robot

Schematics of a tendon-driven continuum robot:

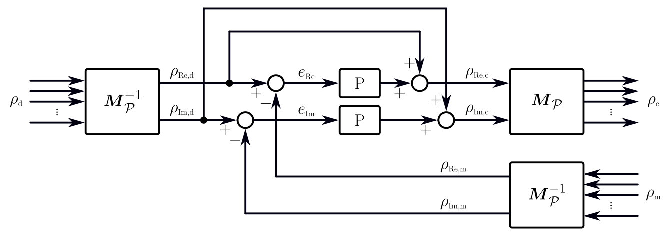



Application to tendon control:




Reducing the design effort and the computational cost by choosing a suitable space to formulate the control problem.

Control schema:




Displacement-control of $n$ displacements using Clarke transform. Both proportional feedback controller with precompensation are sandwich by $\boldsymbol{M}_{\mathcal{P}}$ and $\boldsymbol{M}_{\mathcal{P}}^{-1}$.